Relative Diffeomorphism Group Assembly

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We have previously computed the groups

$$\pi_i \frac{\widetilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)}$$

summarised in the table below:

$*$	**Odd n**	**Even n**
0	0	
1	0	
2	$\mathbb{Z}_2$	
3	$\mathbb{Z}_2$	
4	$\mathbb{Z}$ extended by $\mathbb{Z}_2$	A group of order 2 or 4
5	A group of order 2 or 4	$\mathbb{Z}_2$ or 0.
6	A group of order $1, 2$ or 4	A group of order $2, 4$ or 8

This is all that we could get out of the Hatcher spectral sequence without further input. Here we want to see if we can get anything more using some other known results.

A result of Weiss-Williams [WW, §6.6], pointed out in [Wan24], implies that in the case that for $n = 4k \geq 8$ we have that the map in the LES for the fibration $\mathrm{Diff}_\partial \to \widetilde{\mathrm{Diff}}_\partial \to \frac{\widetilde{\mathrm{Diff}}_\partial}{\mathrm{Diff}_\partial}$ is surjective

$$\pi_2 \widetilde{\mathrm{Diff}}_\partial(D^n) \to \pi_2 \frac{\widetilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)}.$$

Recall by [ABK72, 2.3.3] we have the following commuting diagram for any i, n

$$\begin{array}{ccccc}
\pi_i \mathrm{Diff}_\partial(D^n) & \xrightarrow{\pi_*\iota} & \pi_i \widetilde{\mathrm{Diff}}_\partial(D^n) & \longrightarrow & \pi_i \frac{\widetilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)} \\
\downarrow \gamma & & \downarrow \tilde{\gamma} & & \downarrow ? \\
\pi_{i-1} \mathrm{Diff}_\partial(D^{n+1}) & \xrightarrow{\pi_*\iota} & \pi_{i-1} \widetilde{\mathrm{Diff}}_\partial(D^{n+1}) & \longrightarrow & \pi_{i-1} \frac{\widetilde{\mathrm{Diff}}_\partial(D^{n+1})}{\mathrm{Diff}_\partial(D^{n+1})} \\
\downarrow \gamma & & \uparrow \sim & & \\
\Gamma^{n+i+1}_{(n+1)} & \hookrightarrow & \Theta_{n+i+1} & &
\end{array}$$

where γ is the Gromoll assembly map and ι is the inclusion. In particular the kernel of the map to the relative group is the Gromoll subgroup. Because if something doesnt pull back to step j then it cannot pullback to step $j' > j$ we can see that if the map $\pi_j \widetilde{\mathrm{Diff}}_\partial(D^n) \to \pi_j \frac{\widetilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)}$ has a non-trivial kernel, something that doesnt pull back, then so will the map $\pi_{j'} \widetilde{\mathrm{Diff}}_\partial(D^n) \to \pi_{j'} \frac{\widetilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)}$. Thus we have seen that the relative group is a sort of "total obstruction" for the Gromoll filtration.

Lemma. *There is a map*

$$\pi_i \frac{\widetilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)} \to \pi_{i-1} \frac{\widetilde{\mathrm{Diff}}_\partial(D^{n+1})}{\mathrm{Diff}_\partial(D^{n+1})}$$

making the above diagram commute.

Proof. Because looping preserves fibrations we have a fibration

$$\Omega \mathrm{Diff}_\partial \to \Omega \widetilde{\mathrm{Diff}}_\partial \to \Omega \frac{\widetilde{\mathrm{Diff}}_\partial}{\mathrm{Diff}_\partial}$$

but the cofiber of the first map is given by $\Omega \mathrm{Diff}_\partial / \Omega \widetilde{\mathrm{Diff}}_\partial$ and hence we have a weak equivalence We have actually only shown that their homotopy groups are iso not that there is a weak equiv...

$$\Omega \mathrm{Diff}_\partial / \Omega \widetilde{\mathrm{Diff}}_\partial \simeq \Omega \frac{\widetilde{\mathrm{Diff}}_\partial}{\mathrm{Diff}_\partial}.$$

On the other hand we can assemble the top and bottom to produce a map

$$\pi_i \frac{\Omega \mathrm{Diff}_\partial}{\Omega \widetilde{\mathrm{Diff}}_\partial} \to \pi_{i-1} \frac{\mathrm{Diff}_\partial}{\widetilde{\mathrm{Diff}}_\partial}$$

which we hope makes the diagram commute because it is also just assembly, hard to check without the first part fully understood.

Using this map we can refrase what we said above as the fact that

$$\ker\left(\pi_{i-1} \widetilde{\mathrm{Diff}}_\partial(D^n) \to \pi_{i-1} \frac{\widetilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)}\right) \neq 0 \implies \ker\left(\pi_i \frac{\widetilde{\mathrm{Diff}}_\partial(D^{n-1})}{\mathrm{Diff}_\partial(D^{n-1})} \to \pi_{i-1} \frac{\widetilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)}\right) \neq 0.$$

which is useful when paired with the similar fact

$$\ker\left(\pi_{i-1} \widetilde{\mathrm{Diff}}_\partial(D^n) \to \pi_{i-1} \frac{\widetilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)}\right) \neq 0 \implies \ker\left(\pi_i \widetilde{\mathrm{Diff}}_\partial(D^{n-1}) \to \pi_i \frac{\widetilde{\mathrm{Diff}}_\partial(D^{n-1})}{\mathrm{Diff}_\partial(D^{n-1})}\right) \neq 0.$$

References

[ABK72] P. L. Antonelli, D. Burghelea, and P. J. Kahn. The non-finite homotopy type of some diffeomorphism groups. *Topology*, 11(1):1–49, January 1972.

[Wan24] Wei Wang. A short note on $_1\mathrm{Diff}_{\partial} (D^{4k})$ for $k\geq 3$, April 2024. arXiv:2307.03417 [math].

[WW] Michael S Weiss and Bruce Williams. AUTOMORPHISMS OF MANIFOLDS.